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Instability of Solutions of Impulsive Systems of Differential Equations

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We consider instability of the zero solution of impulsive systems of differential equations with unfixed moments of the impulse effect. Piecewise continuous auxiliary functions are used which are analogues of the classical Lyapunov functions. An analogue of Chetaev's theorem for systems of differential equations without impulses is proved and some corollaries are deduced.

1. INTRODUCTION

In recent years the theory of impulsive systems of differential equations has been elaborated intensively in relation to various applications in biology, mechanics, radioelectronics, control theory, etc. These systems provide the basis for the mathematical simulation of numerous processes and phenomena characterized by the fact that at certain moments of their evolution they undergo short-duration, instantaneous changes. Moreover, the mathematical theory of impulsive systems of differential equations is much richer in problems than the corresponding theory of ordinary differential equations. Among the numerous relevant publications we mention the monographs by Bainov and Simeonov (1989), Lakshmikantham *et al.* (1989), and Samoilenko and Perestyuk (1987).

The use of impulsive systems of differential equations for the mathematical simulation of real processes and phenomena requires the formulation of criteria for the stability and instability of their solutions. Such criteria have been obtained in numerous papers and monographs (Bainov and Simeonov,

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1989; Lakshmikantham *et al.*, 1989; Samoilenko and Perestyuk, 1987; Kulev and Bainov, 1988, 1989a,b).

In the present paper Lyapunov's direct method is applied to the investigation of the instability of the zero solution of impulsive systems of differential equations. To this end piecewise-continuous auxiliary functions are used which are analogues of the classical Lyapunov functions (Kulev and Bainov, 1988, 1989a,b). An analogue of Chetaev's (1934) theorem on the instability of the zero solution of systems of differential equations without impulses is proved, and as a corollary a theorem on instability is deduced considering a linear approximation for impulsive systems.

2. PRELIMINARY NOTES AND DEFINITIONS

Let $\mathbb{R}^n$ be the n-dimensional real space with an arbitrary norm $|\cdot|$, Ω be a domain (an open and connected set) in $\mathbb{R}^n$ containing the coordinate origin, and let $\mathbb{R}_+ = [0, \infty)$.

Consider the following impulsive system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} = f(t, x), & t \neq \tau_k(x), \quad k = 1, 2, \ldots \\ \Delta x|_{t=\tau_k(x)} = I_k(x), & k = 1, 2, \ldots \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, f: $\mathbb{R}_+ \times \Omega \to \mathbb{R}^n$, τ_k: $\Omega \to \mathbb{R}_+$, I_k: $\Omega \to \mathbb{R}^n$, $\Delta x|_{t=\tau_k(x)} = x(t + 0) - x(t - 0)$, $k = 1, 2, \ldots$.

Impulsive systems of the form (1) are characterized by the fact that under the action of an instantaneous force (impact, impulse) the mapping point $(t, x(t))$ of the extended phase space $\mathbb{R}_+ \times \Omega$ on meeting some of the hypersurfaces

$$\sigma_k = \{(t, x) \in \mathbb{R}_+ \times \Omega : t = \tau_k(x)\}, \qquad k = 1, 2, \ldots \tag{2}$$

is instantaneously transferred from the position $(t, x(t))$ into the position $(t, x(t) + I_k(x(t)))$.

Impulsive systems of differential equations of the form (1) are described in detail in Bainov and Simeonov (1989), Lakshmikantham *et al.* (1987), and Samoilenko and Perestyuk (1987).

Let $(t_0, x_0) \in \mathbb{R}_+ \times \Omega$. Denote by $x(t; t_0, x_0)$ the solution of the system (1) which satisfies the initial condition $x(t_0 + 0; t_0, x_0) = x_0$ and by $\mathcal{J}^+ = \mathcal{J}^+(t_0, x_0)$ denote the maximal interval of the form (t_0, ω) in which the solution $x(t; t_0, x_0)$ is defined.

We shall say that conditions (A) are satisfied if the following conditions hold:

A1. $f \in C[\mathbb{R}_+ \times \Omega, \mathbb{R}^n]$ and $f(t, 0) = 0$, $t \in \mathbb{R}_+$.

A2. $I_k \in C[\Omega, \mathbb{R}^n]$ and $I_k(0) = 0$, $k = 1, 2, \ldots$.

A3. $\tau_k \in C[\Omega, \mathbb{R}_+]$, $k = 1, 2, \ldots$, and for each $x \in \Omega$ the relations

$$0 < \tau_1(x) < \tau_2(x) < \cdots < \tau_k(x) < \cdots \qquad \text{and} \qquad \lim_{k\to\infty} \tau_k(x) = \infty$$

are valid.

A4. If D is an open subset of Ω and $x(t; t_0, x_0) \in D$ for each $t \in \mathcal{J}^+$, then $\mathcal{J}^+ = (t_0, \infty)$.

A5. The integral curve of each solution of system (1) intersects each one of the hypersurfaces (2) at most once.

Definition 1. The zero solution of system (1) is said to be *unstable* if

$$(\exists\epsilon > 0)(\exists t_0 \in \mathbb{R}_+)(\forall\delta > 0)(\exists x_0 \in \Omega, |x_0| < \delta)(\exists t^0 \in \mathcal{J}^+(t_0, x_0)):$$

$$|x(t^0; t_0, x_0)| \geqq \epsilon.$$

In our further considerations we shall use the class $\mathcal{V}_0[\mathbb{R}_+ \times \Omega, \mathbb{R}_+]$ of piecewise-continuous auxiliary functions V: $\mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ which are analogues of the classical Lyapunov functions (Kulev and Bainov, 1988, 1989a,b).

Let $\tau_0(x) = 0$ for each $x \in \Omega$ and let

$$G_k = \{(t, x) \in \mathbb{R}_+ \times \Omega: \tau_{k-1}(x) < t < \tau_k(x)\}, \quad k = 1, 2, \ldots; \qquad G = \bigcup_{k=1}^{\infty} G_k$$

Definition 2. We shall say that the function V: $\mathbb{R}_+ \times \Omega \to \mathbb{R}_+$, $(t, x) \to V(t, x)$ belongs to the class $\mathcal{V}_0[\mathbb{R}_+ \times \Omega, \mathbb{R}_+]$ if the following conditions hold:

1. The function V is continuous in G and locally Lipschitz continuous with respect to x in each of the sets G_k, $k = 1, 2, \ldots$.
2. $V(t, 0) = 0$ for $t \in \mathbb{R}_+$.
3. For each $k = 1, 2, \ldots$ and each point $(t_0, x_0) \in \sigma_k$ there exist the finite limits

$$V(t_0 - 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in G_k}} V(t, x), \qquad V(t_0 + 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in G_{k+1}}} V(t, x)$$

Let $V \in \mathcal{V}_0[\mathbb{R}_+ \times \Omega, \mathbb{R}_+]$. For $(t, x) \in G$ set

$$\dot V_{(1)}(t, x) = \lim_{h\to 0^+} \sup h^{-1}[V(t + h, x + hf(t, x)) - V(t, x)]$$

From condition 1 of Definition 2 it follows that if $x = x(t)$ is a solution of system (1), then for $t \neq \tau_k(x)$, $k = 1, 2, \ldots$, the following equality is valid:

$$\dot V_{(1)}(t, x(t)) = \lim_{h\to 0^+} \sup h^{-1}[V(t + h, x(t + h)) - V(t, x(t))]$$

Denote by $\mathcal{K}$ the class of all continuous and strictly increasing functions a: $\mathbb{R}_+ \to \mathbb{R}_+$ and such that $a(0) = 0$.

Introduce the notation $B_\epsilon = \{x \in R^n: |x| < \epsilon\}$, $\overline{B}_\epsilon = \{x \in R^n: |x| \leqq \epsilon\}$, $\epsilon > 0$.

3. MAIN RESULTS

Theorem 1. Let conditions (A) hold and let there exist $t_0 \in R_+$, positive numbers ϵ and L, an open set $D \subset \Omega$, a function $V \in \mathcal{V}_0[[t_0, \infty) \times (D \cup B_\epsilon), R_+]$, and a function $a \in \mathcal{K}$ such that:

(i) $\overline{B}_\epsilon \subset \Omega$.

(ii) $V(t_0 + 0, x) > 0$, $x \in D$.

(iii) $0 < V(t, x) < L$, $t > t_0$, $x \in D$.

(iv) $\dot{V}_{(1)}(t, x) \geqq a(V(t, x))$, $t \in [t_0, \infty)$, $x \in D$, $t \neq \tau_k(x)$, $k = 1, 2, \ldots$.

(v) $V(\tau_k(x) + 0, x + I_k(x)) \geqq V(\tau_k(x), x)$, $x \in D$, $k = 1, 2, \ldots$.

(vi) The boundary ∂D of the set D contains the coordinate origin of R^n.

(vii) $V(t, x) = 0$, $t > t_0$, $x \in \partial D \cap B_\epsilon$.

(viii) $V(\tau_k(x) + 0, x + I_k(x)) = 0$, $x \in \partial D \cap B_\epsilon$, $k = 1, 2, \ldots$.

Then the zero solution of system (1) is unstable.

Proof. Let $\delta > 0$. From conditions (ii) and (vi) of Theorem 1 it follows that there exists $x_0 \in D \cap B_\delta$ such that $V(t_0 + 0, x_0) > 0$. Denote by t_k ($t_k > t_0$), $k = 1, 2, \ldots$, the moments at which the integral curve $\{(t, x(t)): t \in \mathcal{J}^+(t_0, x_0)\}$ of the solution $x(t) = x(t; t_0, x_0)$ intersect the hypersurfaces (2), i.e., $t_k = \tau_k(x(t_k))$, $k = 1, 2, \ldots$.

Let $t \in \mathcal{J}^+$ and suppose, for the sake of definiteness, that $t_m < t \leqq t_{m+1}$ for some positive integer m. If $x(s), x(s + 0) \in D$ for $t_0 < s \leqq t$, then from (v) it follows that

$$\begin{aligned}
&\int_{t_0}^{t} \dot{V}_{(1)}(s, x(s))\, ds \\
&= \int_{t_0}^{t_1} \dot{V}_{(1)}(s, x(s))\, ds + \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} \dot{V}_{(1)}(s, x(s))\, ds \\
&\quad + \int_{t_m}^{t} \dot{V}_{(1)}(s, x(s))\, ds \\
&= V(t_1, x(t_1)) - V(t_0 + 0, x_0) \\
&\quad + \sum_{j=1}^{m-1} [V(t_{j+1}, x(t_{j+1})) - V(t_j + 0, x(t_j + 0))] \\
&\quad + V(t, x(t)) - V(t_m + 0, x(t_m + 0))
\end{aligned}$$

$$
\begin{aligned}
&= V(t_1, x(t_1)) - V(t_0 + 0, x_0) + \sum_{j=1}^{m-1} [V(t_{j+1}, x(t_{j+1})) \\
&\quad - V(t_j + 0, x(t_j) + I_j(x(t_j)))] \\
&\quad + V(t, x(t)) - V(t_m + 0, x(t_m) + I_m(x(t_m))) \\
&\geq V(t_1, x(t_1)) - V(t_0 + 0, x_0) \\
&\quad + \sum_{j=1}^{m-1} [V(t_{j+1}, x(t_{j+1})) - V(t_j, x(t_j))] + V(t, x(t)) - V(t_m, x(t_m)) \qquad (3) \\
&= V(t, x(t)) - V(t_0 + 0, x_0)
\end{aligned}
$$

Moreover, from (iv) and (v) it follows that the function $V(t, x(t))$ is nondecreasing in the interval $\mathcal{J}^+(t_0, x_0)$.

Suppose that for each $t \in \mathcal{J}^+(t_0, x_0)$ we have $x(t) \in D$. Then from condition A4 it follows that $\mathcal{J}^+(t_0, x_0) = (t_0, \infty)$ and from (iii), (3), (iv), and (v) we obtain that the following inequalities are valid:

$$
\begin{aligned}
L &\geqq V(t, x(t)) \\
&\geqq V(t_0 + 0, x_0) + \int_{t_0}^{t} \dot{V}_{(1)}(s, x(s))\, ds \\
&\geqq V(t_0 + 0, x_0) + \int_{t_0}^{t} a(V(s, x(s)))\, ds \\
&\geqq V(t_0 + 0, x_0) + a(V(t_0 + 0, x_0))(t - t_0) \to \infty \qquad \text{as} \quad t \to \infty
\end{aligned}
$$

The contradiction obtained shows that there exists $t^* > t_0$ such that $x(t^*) \overline{\in} D$. Then from (vii) and (viii) it follows that there exists t^0, $t_0 < t^0 \leqq t^*$ such that $|x(t^0)| \geqq \epsilon$. ■

Corollary 1. Let conditions (A) hold and let there exist $t_0 \in \mathbb{R}_+$, a positive number ϵ, an open set $D \subset \Omega$, a function $V \in \mathcal{V}_0[[t_0, \infty) \times (D \cup B_\epsilon), \mathbb{R}_+]$, and functions $a, b \in \mathcal{K}$ such that:

(ia) Conditions (i), (ii), and (v)–(viii) of Theorem 1 hold.

(iia) $0 < V(t, x) \leqq a(|x|)$, $t > t_0$, $x \in D$.

(iiia) $\dot{V}_{(1)}(t, x) \geqq b(|x|)$, $t > t_0$, $x \in D$, $t \neq \tau_k(x)$, $k = 1, 2, \ldots$.

Then the zero solution of system (1) is unstable.

Corollary 2. Let conditions (A) hold and let there exist $t_0 \in \mathbb{R}_+$, positive numbers ϵ and c, an open set $D \subset \Omega$, a function $V \in \mathcal{V}_0[[t_0, \infty) \times (D \cup B_\epsilon), \mathbb{R}_+]$, a function $W \in C[[t_0, \infty) \times D, \mathbb{R}_+]$, and functions $a, b \in \mathcal{K}$ such that:

(ib) Conditions (ia) and (iia) of Corollary 1 hold.

(iib) $\dot{V}_{(1)}(t, x) \leqq cV(t, x) + W(t, x)$, $t > t_0$, $x \in D$, $t \neq \tau_k(x)$, $k = 1, 2, \ldots$.

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